Online Algorithms for 1-Space Bounded 2-Dimensional Bin Packing and Square Packing

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Abstract. In this paper, we study 1-space bounded 2-dimensional bin packing and square packing. A sequence of rectangular items (square items) arrive one by one, each item must be packed into a square bin of unit size on its arrival without any information about future items. When packing items, $90°$-rotation is allowed. 1-space bounded means there is only one "active" bin. If the "active" bin cannot accommodate the coming item, it will be closed and a new bin will be opened. The objective is to minimize the total number of bins used for packing all items in the sequence.
Our contributions are as follows: For 1-space bounded 2-dimensional bin packing, we propose an online packing strategy with competitive ratio 5.06. The lower bound of the competitive ratio is proved to be 3.17. Moreover, we study 1-space bounded square packing, where each item is a square with side length no more than 1. A 4.3-competitive algorithm can be achieved, and the lower bound of the competitive ratio is shown to be 2.94. All these bounds surpass the previous results.

1 Introduction

Bin packing is one of the most fundamental problems in computer science. In the online fashion of bin packing, a sequence of items arrive over time, each item must be packed into a bin on its arrival without any information about future items. The objective is minimizing the number of used bins for packing all items in the sequence.

Roughly speaking, the problem of online bin packing has two models: the *unbounded space* model and the *bounded space* model. In the unbounded space model, any bin can be used to pack the coming item if its empty space is large enough. In the bounded space bin packing, only "active" bins can be used to pack item, and the number of "active" bins is bounded by some constant. If all "active" bins cannot accommodate the coming item, one of such bins will be closed and a new bin will be opened to pack the coming item.

Our focus in this paper is the bounded space 2-dimensional bin packing, and the number of "active" bins is restricted to be one. We call it *1-space bounded 2-dimensional bin packing*. In this variant, the coming item is packed either in the "active" bin, or in a new "active" bin after the closure of the previous "active" bin. The closed bin cannot be used to pack items any more. In 1-space bounded bin packing, 90° rotation on item is allowed, otherwise, the performance ratio will be unbounded [8]. We also consider 1-space bounded square packing, where each item is a square with side length no more than 1. Our target is to find packing strategies for 1-space bounded 2-dimensional bin packing and square packing to minimize the number of used bins.

To measure the performance of the 1-space bounded 2-dimensional bin packing, we use the asymptotic competitive analysis, which is often used for online problems. For a sequence σ of items, let $A(\sigma)$ and $OPT(\sigma)$ denote the number of used bins by the online packing strategy A and the offline optimal algorithm OPT, respectively. The *asymptotic competitive ratio* of the online algorithm A is defined to be

$$R_A^\infty = \lim_{k\to\infty} \sup_\sigma \{\frac{A(\sigma)}{OPT(\sigma)} | OPT(\sigma) = k\}.$$

Related works:

The online bin packing has been studied for more than thirty years. For one-dimensional online bin packing, Johnson et al. [12] showed that the First Fit algorithm (FF) has an asymptotic competitive ratio of 1.7. Yao [17] improved the algorithm to obtain a better upper bound of 5/3. Lee et al. [13] introduced the class of Harmonic algorithms, and showed that an asymptotic competitive ratio of 1.63597 is achievable. The best known upper bound is 1.58889, which was given by Seiden [14]. As for the lower bound of the competitive ratio of one dimensional bin packing, Yao [17] showed that no online algorithm can have an asymptotic competitive ratio less than 1.5. The best known lower bound is 1.54014 [16]. For two-dimensional online bin packing, Seiden and van Stee [15] showed an upper bound of 2.66013 by implementing the Super Harmonic Algorithm. The best known upper bound of the competitive ratio for two dimensional bin packing is 2.5545, which was given by Han et al. [9]. The best known lower bound is 1.907 [1].

For bounded space bin packing, Harmonic algorithm by Lee et al. [13] can be applied for one dimensional case, the competitive ratio is 1.69103 when the number of active bins goes to infinity. Csirik and Johnson [3] presented an 1.7-competitive algorithm (K-Bounded Best Fit algorithms (BBF_K)) for one dimensional bin packing using K active bins, where $K \geq 2$. For multi-dimensional case, Epstein et al. [5] gave a 1.69103^d-competitive algorithm using $(2M-1)^d$ active bins, where $M \geq 10$ is an integer such that $M \geq 1/(1-(1-\varepsilon)^{1/(d+2)})-1$, $\varepsilon > 0$ and d is the dimension of the bin packing problem. For *1-space bounded 2-dimensional bin packing*, Fujita [8] first gave an $O((\log\log m)^2)$-competitive algorithm, where m is the width of the square bin and the size of each item is

$a \times b$ (a, b are integers and $a, b \le m$). Chin et al. proposed an 8.84-competitive packing strategy [4], and the upper bound was further improved to be 5.155 [18], they also gave the lower bound 3 for 1-space bounded two dimensional bin packing. If the item is restricted to be square, Zhang et al. [18] showed that the upper bound and lower bound of the competitive ratio are 4.5 and 8/3, respectively. For 1-space bounded d-dimensional bin packing (d can be any integer), a 4^d-competitive packing strategy was given in [19].

In the remaining part, 1-space bounded 2-dimensional bin packing is studied in Section 2, both upper and lower bounds of the competitive ratio are given; and in Section 3, we consider 1-space bounded square packing by showing the upper and lower bounds.

2 1-Space Bounded 2-Dimensional Bin Packing

2.1 Upper bound

In this section, we give a 5.06-competitive algorithm for 1-space bounded 2-dimensional bin packing, improving the previous upper bound 5.15. Since 90°-rotation is allowed, we may assume that for each rectangular item (w, h), the width is no less than the height, i.e., $w \ge h$.

We classify the rectangular items into three classes A, B and C according to the width w:

$$A = \{(w, h) | w > 1/2\},$$
$$B = \{(w, h) | 1/8 < x \le 1/2\}, and$$
$$C = \{(w, h) | w \le 1/8\}.$$

For simplicity, let A-item denote an item belonging to class A. B-item and C-item are defined similarly.

Since all items are rectangular, in the packing strategy, items are packed with sides either vertical or parallel to the boundary of the bin. For A-items, the packing strategy pack them in the active bin using a top-down approach starting from the upper boundary of the bin. Since the width of any A-item is strictly larger than 1/2, any two A-items cannot share the same horizontal line within a bin. The width of B-item and C-item are upper bounded by 1/2, in the packing strategy, they are packed either in the left half side or in the right half side of the bin by using a bottom-up approach starting from the lower boundary of the bin. The heights of the left side and the right side are packed as balance as possible. If an item cannot be packed into the bin using the strategy, the active bin will be closed and a new one will be opened to pack this item.

Let the *occupation ratio* to be the utilization of some area in the bin. For an A-item (w_1, h_1), since no other items share the same horizontal line in the bin, the strip $(1, h_1)$ containing this A-item cannot be used to pack other item, thus, the *occupation ratio* of this strip is $\frac{w_1 \cdot h_1}{1 \cdot h_1}$. Since the width of any A-item is larger than 1/2, the occupation ratio of any strip containing A-item is strictly larger than 1/2. For an B-item (w_2, h_2), since it is packed into either the left side or the right side, the strip $(1/2, h_2)$ containing this B-item cannot be used to pack other item, thus, the occupation ratio of this strip is $\frac{w_2 \cdot h_2}{1/2 \cdot h_2}$. Since the width of B-item is in between 1/8 and 1/2, the occupation ratio of any strip containing B-item is at least 1/4. Note that C-item may be very tiny, if it is packed by using the same way as for A-item or B-item, the wastage will be very large and the performance will be bad.

We first consider how to pack C-items. The width of C-item is upper bounded by 1/8, but it is not lower bounded by any positive value. C-items are classified into subclasses $C_1, C_2, C_3, ...$ according to their widths. Let $c_1 = 1/8$, $c_i = a \cdot c_{i-1}$ for $i > 1$, where $a = (6 - \sqrt{21})/15 < 1/9$. We say

$$\text{an item } (w, h) \text{ belongs to subclass} \begin{cases} C_{2i-1} & \text{if } 3a \cdot c_i < w \leq c_i \\ C_{2i} & \text{if } a \cdot c_i < w \leq 3a \cdot c_i \end{cases}$$

Thus, c_i is the maximal width of items from subclass C_{2i-1} and $3a \cdot c_i$ is the maximal width of items from subclass C_{2i}. Each item belonging to subclass C_{2i-1} and C_{2i} $(i > 0)$ can be packed into a row with height c_i and width 1/2. The items from subclasses C_{2i-1} are packed from left to right while the items from subclass C_{2i} are packed from right to left in three subrows (upper, middle and lower), keeping the lengths of these three subrows balanced at all times (that means a new item is always packed into the subrow with the shortest packed length). Note that C-items are packed with a 90°-rotation. Figure 1 depicts a row with packed items from subclass C_{2i-1} and C_{2i}. When handling an item from subclass C_{2i-1} (or C_{2i}), a new row of height c_i will be created if the existing rows with height c_i cannot accommodate this item by the above packing method.

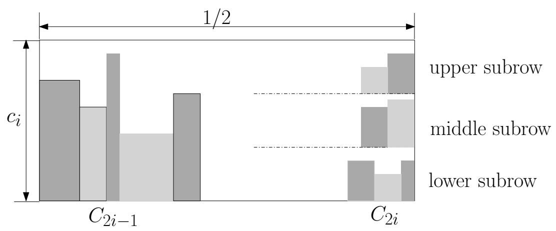

Fig. 1. Packing C_{2i-1} (or C_{2i})-items $(i > 0)$ into a row.

Consider the packing of C-items. If there are more than one rows for the subclass C_{2i-1} and C_{2i}, the last row could be almost empty and the non-last rows are almost full. The total height of the last rows is at most

$$\sum_{i>0} c_i = \frac{c_1}{1-a} = \frac{1/8}{1-(6-\sqrt{21})/15} \approx 0.138.$$

Now we analyze the occupation ratio of the non-last rows for subclass C_{2i-1} and C_{2i}. In the left side, suppose the occupied length is x, thus, the total occupation in the left side is at least

$$3a \cdot c_i \cdot x.$$

In the right side, suppose the length of the longest occupied subrow is y_1 and the length of the shortest occupied subrow is y_2, the total occupation in the right side is at least

$$3a \cdot c_i \cdot y_2 + a \cdot c_i \cdot (y_1 - y_2).$$

Since $y_1 - y_2 \leq 3a \cdot c_i$ (that is because the balanced packing in the right side), the above formula is at least

$$3a \cdot c_i \cdot y_1 - 6a^2 \cdot c_i^2.$$

In the left side, the height of the item may be larger than $3a \cdot c_i$. Using an amortized analysis as follows, if packing an item (w, h) which belongs to subclass C_{2i-1} will create a new row, this item contributes $\max\{0, h^2 - 3a \cdot c_i \cdot h\}$ to the row which cannot pack it. If $h > 3a \cdot c_i$, the contribution of this item in the newly created row is $3a \cdot c_i \cdot h$ and the remaining area of this item is larger than $h^2 - 3a \cdot c_i \cdot h$, which will contribute to the previous row.

Lemma 1. *For any non-last row with height c_i ($i > 0$), the amortized occupation ratio is at least 1/4. (The proof is in Appendix.)*

The packing strategy can be described as follows.

Algorithm Packing-Bin: for 1-space bounded 2-dimensional bin packing

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1: A-items are packed in a top-down order starting from the top boundary of the bin.
2: B-items and C-items are packed in a bottom-up order along both the left and right side of the square bin, keeping the heights of these two sides balanced at all times, i.e., a new B-item or newly created row of C-item is always packed on the side with smaller height.
3: If there is insufficient space to pack a new item (A-item, B-item) or create a new row for the coming C-item, the bin is closed and a new bin is opened to pack the new item or row.

An example of a packing configuration by applying the algorithm Packing-Bin is illustrated in Figure 2. In this configuration, the height of the packed A-items is y, the left and right sides of the packed B-items and C-items are of height y_1 and y_2 respectively. In this example, $y_1 > y_2$, according to the algorithm, if a B-item comes, we pack it in the right side.

Now we give the competitive ratio of the algorithm Packing-Bin.

Theorem 1. *The competitive ratio of the packing strategy **Packing-Bin** is at most 5.06. (The proof is in Appendix.)*

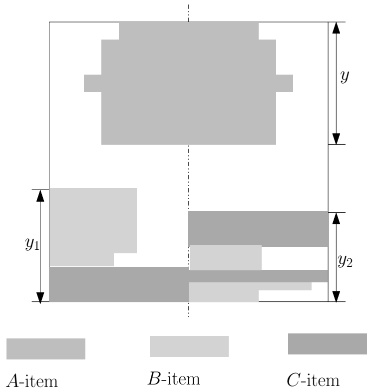

Fig. 2. Packing rectangular items into a square bin.

2.2 Lower bound

In this part, we prove that the lower bound of the competitive ratio is at least 3.167, improving the previous bound 3.

Theorem 2. *The lower bound of the competitive ratio for 1-space bounded 2-dimensional bin packing is at least 3.167.*

Proof. Consider a sequence of items: $S = \{X_1, X_2, ..., X_{2n}, A_1, B_1, A_2, B_2, ..., A_n, B_n, T_1, T_2, ..., T_n\}$.

In the first part of the item sequence containing all the X_i items,

$$\begin{aligned} X_{2i-1} &= (1/2 + i \cdot \epsilon, 1/2 + i \cdot \epsilon) \\ X_{2i} &= (1/2 - (i-1) \cdot \epsilon, 1/2 - (i-1) \cdot \epsilon) \end{aligned}$$

in which $\epsilon = o(1/n^2)$. It can be verified that no online algorithm can pack any two consecutive items into one unit square bin because the sum of the edge lengths of any two consecutive X-items is larger than 1. Thus, at least $2n$ bins are used for packing all these X_i items. However X_{2i-1} and X_{2i+2} can be packed into the same bin in the optimal packing.

In the second part of the item sequence containing all A_i and B_i items,

$$\begin{aligned} A_i &= (1/3 + \epsilon, 2/3 + \epsilon_{i+1}) \\ B_i &= (1/3 + \epsilon, 1/3 - \epsilon_i) \end{aligned}$$

in which $\epsilon = o(1/n^2)$, $\epsilon_i < \epsilon_{i+1}$ for $i \geq 1$, and $0 < \epsilon_i = o(1/n^2)$ for all $i \geq 1$. In the online fashion, A_i and A_{i+1} cannot be packed into the same bin. Thus, no online algorithm can pack the second part of the item sequence by using less than n bins. However X_{2i-1}, X_{2i+2}, A_i, B_{i+1} can be packed into one bin in the optimal packing.

In the third part of the item sequence containing all T_i items,

$$T_i = (1, 1/7 + \epsilon)$$

in which $\epsilon = o(1/n^2)$. It can be verified that any bin can contain at most 6 items from this part. However, one T_i can be packed together with X_{2i-1}, X_{2i+2}, A_i, B_{i+1} in the optimal packing, as shown in Figure. 3.

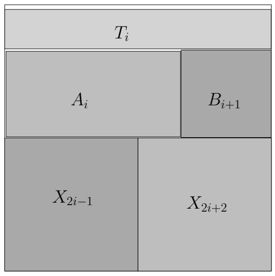

Fig. 3. The optimal packing

Combine these three parts, note that our focus is the asymptotic performance, there is no online algorithm which can pack all items in this sequence within $2n+n+n/6$ bins, while the optimal strategy only uses n bins. Thus, we conclude that no online algorithm can achieve a competitive ratio less than 3.167 for 1-space bounded 2-dimensional bin packing. □

3 1-Space Bounded Square Packing

In 1-space bounded square packing, the arrival items are squares with side length no more than 1. In our packing strategy, square items are packed in bricks where a *brick* is a rectangle with aspect ratio $\sqrt{2}$. Packing square items in bricks is a popular used method. The interesting property of this method is: a brick can be partitioned into two smaller congruent bricks of the same size. Thus, packing a square into a brick can be done recursively. Given a square Q, let $S(Q)$ denote the smallest brick which can contain Q. Let $|R|$ denote the area of rectangle R.

The following is a modified algorithm similar to [11] for packing a square Q in a brick T.

Algorithm Brick(Q,T): Packing a square Q in brick T

1: If there is no empty brick in T of size greater than or equal to $S(Q)$, then give up packing Q in T.
2: Else pack Q in T as follows:
  – if there is an empty brick congruent to $S(Q)$, then pack Q into it;
  – else partition the smallest empty brick P that is larger than $S(Q)$ into two congruent bricks P_1 and P_2. Assume P_1 is the left (or the upper) one. Recursively execute $Brick(Q, P_1)$.

Lemma 2. *[11] If the above algorithm cannot pack an item Q in a brick B, then all empty bricks in B are smaller than $S(Q)$. Furthermore, there is at most one empty brick with area $|S(Q)|/2^i$ for each $i = 1, 2, ...$, and the total area of the empty bricks is less than $|S(Q)|$.*

Lemma 3. *If Q is packed in a brick congruent to $S(Q)$, then at least $1/(2\sqrt{2})$ of this brick is occupied. (The proof is in Appendix.)*

3.1 Upper bound

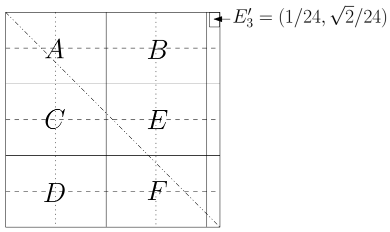

Fig. 4. Partition of unit bin.

We partition each unit bin as shown in Figure 4. Bricks A to F are of the same size $(1/3, \sqrt{2}/3)$, and each brick can be further partitioned into two congruent bricks. We call an item *small*, *middle*, and *large* if the edge length ℓ satisfies $\ell \leq 1/3$, $1/3 < \ell \leq 1/2$, and $\ell > 1/2$, respectively. There is a small brick $E'_3 = (1/24, \sqrt{2}/24)$ in the right-top of the bin. This brick is used only in some special cases, which will be described in later analysis. The packing strategy is described as follows.

Algorithm Packing-Square: For 1-space bounded square packing

1: For a small item s, by using the algorithm **Brick()**, we search A, B, C, D, E, F, in the listed order, for an $S(s)$ to pack s. E.g., if **Brick(s, A)** cannot pack s, then consider **Brick(s, B)**.

2: For a middle item s, we search in the order of (1) the left-bottom corner of the bin; (2) the right-bottom corner of the bin; (3) immediately to the left of the middle item, which is packed on the right-bottom corner of the bin; (4) the right-top corner of the bin; and (5) the left-top corner of the bin to pack item s. ▷ Note that packing a middle item in the left-corner of the bin may overlap with bricks E and F since the side length may be larger than $\sqrt{2}/3$. In this case, bricks E and F are slightly shifted to the right such that there is no overlap with the packed middle item.

3: For a large item s, we pack it at the right-bottom corner of the unit bin.

4: If item s cannot be packed into the active bin by using the above rules, this bin will be closed then a new bin will be opened to pack s.

Now we analyze the competitive ratio of the above algorithm. For a sequence of square items, assume the offline optimal packing strategy uses n bins, and our algorithm uses $x + y + z$ bins, where x is the number of bins containing a large item, y is the number of bins not containing large item and closed by the packing of a large item, and z is the number of remaining bins. In the optimal packing, a bin contains at most one large item, thus, $x \leq n$. From the definition of y, for each bin with large item, there is at most one previous bin counted in those y bins, thus, $y \leq x$.

By using the idea of amortized analysis, if the amortized occupation of those x, y, and z items are a, b, and c, respectively, the total area of the sequence of items is at least $ax + by + cz$, which is upper bounded by n since the optimal packing uses n bins. Therefore, the competitive ratio is

$$\begin{aligned} \max \;\; & (x+y+z)/n \\ \text{s.t.} \;\; & ax+by+cz \leq n \quad \text{and} \\ & y \leq x \leq n \end{aligned} \tag{1}$$

In the amortized analysis, if there is a large item s with side length $\ell > 1/2$ in a bin, this item contributes $\ell^2 - 1/4$ to the previous bin and the remaining area which contributes to its packed bin is $1/4$; if a bin is closed by the packing of a middle item s' with side length $1/3 < \ell' \leq 1/2$, this item contributes $(\ell^2 - 1/9)/2$ to the previous bin, and the remaining area which contributes to its packed bin is $\ell^2 - (\ell^2 - 1/9)/2 = \ell^2/2 + 1/18 \geq 1/9$.

For those x bins containing large items, the remaining area is at least $1/4$. Thus, we may set $a = 1/4$.

For those y bins, they are closed by the packing of large items. The following lemma analyzed the amortized occupation in this part of bins.

Lemma 4. *The amortized occupation in those y bins is at least* $(10 - 6\sqrt{2})/9$. *(The proof is in Appendix.)*

Thus, $b = (10 - 6\sqrt{2})/9 \approx 0.1683$.

Now we give the amortized occupation for the remaining z bins.

Lemma 5. *The amortized occupation in those z bins is at least* $1/4$. *(The proof is in Appendix.)*

Thus, $c = 1/4$.

Next, we give the competitive ratio of the algorithm Packing-Square.

Theorem 3. *The competitive ratio of the algorithm Packing-Square is at most 4.3268.*

Proof. Taking the values of a, b, and c into Formula (1),

$$\begin{aligned} \max \;\; & (x+y+z)/n \\ \text{s.t.} \;\; & x/4 + 0.1683y + z/4 \leq n \quad \text{and} \\ & y \leq x \leq n \end{aligned} \tag{2}$$

Therefore,

$$(x + y + z)/4 \le n + 0.0817y \le 1.0817n.$$

The competitive ratio of the packing strategy is at most

$$\frac{x + y + z}{n} \le 1.0817 \cdot 4 = 4.3268$$

□

3.2 Lower bound

Now we derive a lower bound of the competitive ratio for 1-space bounded square packing. Roughly speaking, the adversary sends items in phases.

- In the first phase, the side lengths of the coming items are very close to 1/2.
- In the second phase, the side lengths of the items are very close to 1/3.
- ...

The high level idea underlying the lower bound proof is as follows: The adversary constructs a sequence with $2n$ items in the first phase, $3n$ items in the second phase, ... such that no online packing algorithm can use less than $2n$ bins for the first phase, $3n/4$ bins for the second phase, ... But for the optimal packing strategy, n bins is sufficient to pack all items.

Theorem 4. *There is no online algorithm with a competitive ratio less than* 2.75 *for 1-space bounded square packing.*

Proof. As mentioned above, the adversary sends items in phases. In the first phase, the item sequence is $(Y_1, X_1, Y_2, X_2, ..., Y_n, X_n)$. Let $\epsilon = o(1/n^2)$. Let y_i and x_i denote the side length of X_i and Y_i, respectively. The side lengths of these items are as follows.

$$y_i = 1/2 - (n + 1 - i)\epsilon$$

$$x_i = 1/2 + (n + 2 - i)\epsilon$$

It can be verified that any two adjacent items cannot be packed into one bin. Thus, no online algorithm can pack these items by using less than $2n$ bins. However, items Y_i and X_{i+1} can be packed into the same bin by the optimal strategy.

In the second phase, the adversary sends $3n$ items, the arrival order is (U_3, U_4, W_1, W_2, U_5, U_6, W_3, W_4, ...U_{n-1}, U_n, W_{n-3}, W_{n-2}, U_1, U_2, V_1, V_2, ...,V_n, W_{n-1}, W_n). Let u_i, v_i, and w_i denote the side length of U_i, V_i, and W_i, respectively. Let $\epsilon \ll \epsilon_1$, $\epsilon_i < \epsilon_{i+1}$ for $i \ge 1$, $\epsilon_{2i+1} > \epsilon_{2i-1} + \epsilon_{2i} + 2\epsilon$ for $i \ge 1$, and $\epsilon_i = o(1/n^2)$ for any i. The side lengths of these items are as follows.

$$u_i = 1/3 + \epsilon_i$$
$$v_i = 1/3 + \epsilon$$
$$w_i = 1/3 - \epsilon - \epsilon_i$$

It can be verified that except (U_n, W_{n-3}, W_{n-2}, U_1, U_2, V_1) and (V_n, V_n, V_n, W_{n-1}, W_n)), any adjacent five items cannot be packed into the same bin. Since we consider the asymptotic performance, i.e., n is very large, no online algorithm can pack these items by using $3n/4$ bins. However, items U_i, V_i, and W_i can be packed together with Y_i and X_{i+1}.

After the second phase, any online algorithm uses at least $2n + 3n/4$ bins, while the optimal packing strategy only uses n bins. Thus, the competitive ratio is at least 2.75. □

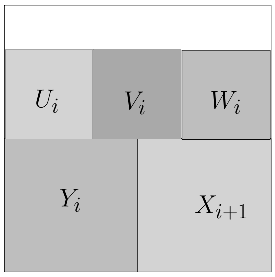

Fig. 5. The optimal packing

For the above item sequence, the optimal packing in a bin is shown in Figure. 5. There are still some free space in the upper part of the optimal packing. We can fully utilize these free space to force the online algorithm uses more bins. In the optimal packing, the height of the empty part is around 1/6. The adversary may design another phase with $7n$ items whose side lengths are around 1/7, such that no consecutive 37 items can be packed into the same bin, thus, $7n/36$ bins are needed for the online packing. In the optimal strategy, 7 item can be packed into the upper part of the optimal packing as shown in Figure. 5. Thus, we have the following claim.

Claim. There is no online algorithm with a competitive ratio less than 2.94 for 1-space bounded square packing.

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Appendix.

A.1 The proof for Lemma 1

Proof. Consider the packing configuration as shown in Figure 1. As mentioned before, x is the length of the left occupied area, y_1 and y_2 are the maximal and minimal lengths of the right occupied subrows.

To show the amortized occupation ratio, we study two cases.

- If this configuration cannot accommodate the next item (w, h) from subclass C_{2i-1}, we have $x + y_1 + h > 1/2$ and $h \le w \le c_i$. The amortized occupation in this row is at least

  $$3a \cdot c_i \cdot x + 3a \cdot c_i \cdot y_1 - 6a^2 \cdot c_i^2 + \max\{0, h^2 - 3a \cdot c_i \cdot h\}.$$

  - If $h \le 3a \cdot c_i$, the amortized occupation is at least

    $$\begin{aligned} &3a \cdot c_i \cdot x + 3a \cdot c_i \cdot y_1 - 6a^2 \cdot c_i^2 \\ &> 3a \cdot c_i \cdot (1/2 - h) - 6a^2 \cdot c_i^2 \\ &\ge 3a \cdot c_i(1/2 - 3a \cdot c_i) - 6a^2 \cdot c_i^2 \\ &= 3/2 \cdot a \cdot c_i - 15a^2 \cdot c_i^2 \end{aligned}$$

    Since the area of this row is $c_i/2$, the amortized occupation ratio in this case is

    $$3a - 30a^2 \cdot c_i \ge 3a - 15a^2/4 = 1/4.$$

  - If $h > 3a \cdot c_i$, the amortized occupation is at least

    $$\begin{aligned} &3a \cdot c_i \cdot x + 3a \cdot c_i \cdot y_1 - 6a^2 \cdot c_i^2 + h^2 - 3a \cdot c_i \cdot h \\ &> 3a \cdot c_i \cdot (1/2 - h) - 6a^2 \cdot c_i^2 + h^2 - 3a \cdot c_i \cdot h \\ &= h^2 - 6a \cdot c_i \cdot h + 3/2 \cdot a \cdot c_i - 6a^2 \cdot c_i^2 \\ &= (h - 3a \cdot c_i)^2 + 3/2 \cdot a \cdot c_i - 15a^2 \cdot c_i^2 \\ &\ge 3/2 \cdot a \cdot c_i - 15a^2 \cdot c_i^2 \end{aligned}$$

    Since the area of this row is $c_i/2$, the amortized occupation ratio in this case is

    $$3a - 30a^2 \cdot c_i \ge 3a - 15a^2/4 = 1/4.$$

- If this configuration cannot accommodate the next item (w, h) from subclass C_{2i}, we have $x + y_2 + h > 1/2$ and $h \le w \le 3a \cdot c_i$. The amortized occupation in this row is at least

  $$3a \cdot c_i \cdot x + 3a \cdot c_i \cdot y_1 - 6a^2 \cdot c_i^2.$$

  From previous analysis, we know the ratio between this value and $c_i/2$ is at least 1/4.

Combine the above two cases, the amortized occupation ratio of the non-last row for C-items is at least 1/4. □

A.2 The proof for Theorem 1

Proof. For a given sequence of items, suppose the number of bins used by the packing strategy Packing-Bin is n. Let o_A^i, o_B^i and o_C^i be the occupied space of A-, B- and C- items in the i-th bin respectively. The average occupation for all the bins is $\sum_{i=1}^{n}(o_A^i + o_B^i + o_C^i)/n$.

Consider the packing configuration of the i-th bin as shown in Figure 2, assume that the height of the packed A-items is y, the left and right sides of the packed B-items and C-items are of height y_1 and y_2 respectively. W.l.o.g., $y_1 \geq y_2$. We have

$$\begin{aligned}
&o_A^i \geq y/2 \\
&o_B^i \geq (y_1 + y_2 - \textstyle\sum_{j\geq 1} c_j - m)/8 \geq (y_1 + y_2 - 0.138 - m)/8 \\
&o_C^i \geq m/8
\end{aligned}$$

where m is the total height of the non-last rows of C-items.

In the following amortized analysis, if an A-item or an B-item cannot be packed into the active bin by the packing algorithm, this item will contribute some area to the just recently closed bin. For the i-th bin, let q_A^i and q_B^i be the contribution of A-items and B-items to the $(i-1)$-th bin, and p_A^i and p_B^i be the remaining areas of A-items and B-items in the i-th bin. Formally, if an A-item (w, h) cannot be packed into the i-th bin,

$$q_A^i = \frac{w \cdot h}{2};$$

if an B-item (w, h) cannot be packed into the i-th bin,

$$q_B^i = \begin{cases} w \cdot h/2 & \text{if } 1/4 \leq h < 1/2 \\ h^2 - h/8 & \text{if } 1/8 \leq h < 1/4 \end{cases}$$

For this item, the remaining area is

$$\begin{cases} w \cdot h/2 \geq h/8 & \text{if } 1/4 \leq h < 1/2 \\ w \cdot h - h^2 + h/8 \geq h/8 & \text{if } 1/8 \leq h < 1/4 \end{cases}$$

Since we focus on the asymptotic performance, when n is very large, we have

$$\frac{\sum_{i=1}^{n}(o_A^i + o_B^i + o_C^i)}{n} \geq \min_{1\leq i<n}\{p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1}\} \tag{3}$$

- If the next A-item with height u cannot be packed into this bin, we have $y + y_1 + u > 1$ and $q_A^{i+1} \geq u/4$. Thus,
  - If $y_1 - y_2 \leq 1/4$, the amortized occupation in this bin is at least

$$\begin{aligned}
&p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1} \\
&\geq y/4 + (y_1 + y_2 - 0.138 - m)/8 + u/4 + m/8 \\
&\geq y/4 + (y_1 + y_2)/8 + u/4 - 0.01725 \\
&> (1 - y_1)/4 + (y_1 + y_2)/8 - 0.01725 \\
&= 0.23275 - (y_1 - y_2)/8 \\
&\geq 0.2015
\end{aligned}$$

- If $y_1 - y_2 > 1/4$, that means the top item in the bottom-left occupied area is an B-item (w', h'), which may be the first item in this bin. Thus, $q_B^i = w' \cdot h'/2$ and $p_B^i = o_B^i - q_B^i \geq (y_1 + y_2 - 0.138 - m - h')/8 + w' \cdot h'/2$. Let $h' = y_1 - y_2 + x$, where $x \geq 0$. Note that $w' \geq h' \geq y_1 - y_2 > 1/4$, we have

$$\begin{aligned}
p_B^i &\geq (y_1 + y_2 - 0.138 - m - h')/8 + w' \cdot h'/2 \\
&= (y_2 + y_2 - 0.138 - m)/8 + w' \cdot (y_1 - y_2)/2 + w' \cdot x/2 - x/8 \\
&> (y_2 + y_2 - 0.138 - m)/8 + (y_1 - y_2)^2/2
\end{aligned}$$

It follows that

$$\begin{aligned}
&p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1} \\
&\geq y/4 + (y_2 + y_2 - 0.138 - m)/8 + (y_1 - y_2)^2/2 + u/4 + m/8 \\
&= (y + u)/4 + y_2/4 - 0.01725 + (y_1 - y_2)^2/2 \\
&> (1 - y_1)/4 + y_2/4 - 0.01725 + (y_1 - y_2)^2/2 \\
&= 0.23275 + (y_1 - y_2)^2/2 - (y_1 - y_2)/4 \\
&\geq 0.2015
\end{aligned}$$

– If the next B-item (w, h) cannot be packed into this bin, we have $y + y_2 + h > 1$.
  - If $h \geq 1/4$, then $q_B^{i+1} = w \cdot h/2 \geq h^2/2$. Thus

$$\begin{aligned}
&p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1} \\
&\geq y/4 + (y_1 + y_2 - 0.138 - m)/8 + h^2/2 + m/8 \\
&= y/4 + (y_1 + y_2)/8 + h^2/2 - 0.01725 \\
&\geq y/4 + y_2/4 + h^2/2 - 0.01725 \\
&> (1 - h)/4 + h^2/2 - 0.01725 \\
&\geq 0.2015
\end{aligned}$$

  - If $h < 1/4$, then $q_B^{i+1} = h^2 - h/8$. Thus,

$$\begin{aligned}
&p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1} \\
&\geq y/4 + (y_1 + y_2 - 0.138 - m)/8 + m/8 + (h^2 - h/8) \\
&= y/4 + (y_1 + y_2)/8 - 0.01725 + (h^2 - h/8) \\
&\geq y/4 + y_2/4 - 0.01725 + (h^2 - h/8) \\
&> (1 - h)/4 - 0.01725 + (h^2 - h/8) \\
&\geq 0.1976
\end{aligned}$$

– If the next C-item with height u cannot be packed into this bin, as the height of each row of C-item is at most $1/8$, we have $y + y_2 + 1/8 > 1$. Thus,

$$\begin{aligned}
&p_A^i + p_B^i + o_C^i + q_A^{i+1} + q_B^{i+1} \\
&\geq y/4 + (y_1 + y_2 - 0.138 - m)/8 + m/8 \\
&= y/4 + (y_1 + y_2)/8 - 0.01725 \\
&\geq y/4 + y_2/4 - 0.01725 \\
&> (1 - 1/8)/4 - 0.01725 \\
&= 0.2015
\end{aligned}$$

Combining all of the above cases, the amortized occupation in each bin is at least 0.1976, thus, the competitive ratio of this packing strategy is at most $1/0.1976 = 5.06$. □

A.3 The proof for Lemma 3

Proof. Suppose an item Q is packed into a brick $R = (r, r/\sqrt{2})$ which is congruent to $S(Q)$. Since R is the smallest brick to pack Q, the side length of Q is at least $r/2$. Thus, at least $1/(2\sqrt{2})$ of this brick is occupied.

A.4 The proof for Lemma 4

Proof. W.l.o.g., assume that the i-th bin does not contain large item and closed by the packing of a large item s with side length $\ell > 1/2$.

- If $\ell \geq 2/3$, the amortized occupation of the i-th bin is at least $(2/3)^2 - 1/4 = 7/36$.
- Else, $1/2 < \ell < 2/3$ and packing s on the right-bottom corner of the i-th bin will overlap with some packed item in C or D. If packing s on the right-bottom corner of the i-th bin overlaps with a middle item s' with side length $1/3 < \ell' \leq 1/2$ which is packed on the left-bottom corner of the bin, the amortized occupation in this bin is at least

$$\begin{aligned} & {\ell'}^2 - ({\ell'}^2 - 1/9)/2 + \ell^2 - 1/4 \\ = & \ell^2 + {\ell'}^2/2 - 7/36 \\ \geq & \ell^2/2 + (\ell + \ell')^2/4 - 7/36 \\ \geq & 1/8 + 1/4 - 7/36 \\ = & 13/72 \end{aligned}$$

  since $\ell + \ell' > 1$ and s' may lead the close of the previous bin.
- If pack s on the right-bottom corner of the i-th bin overlaps with a middle item which is also packed in the right-bottom corner of this bin. In this case, at least one small item or middle item is packed in C. If a small item is packed in C, the total occupation of small items in this bin is at least $1/9$ since both A and B cannot pack this small item. Thus, the amortized occupation is at least $2/9$.
- Else, if packing s on the right-bottom corner of the bin overlaps with a small item in C or D, the side length ℓ of s satisfies $\ell > 1 - \sqrt{2}/3$. From the packing strategy, the side length ℓ' of the overlapped small item s' is no more than $\sqrt{2}/6$ since $\ell < 2/3$ in this case. Thus, the overlap must happens in the right part of C or D, in another words, A, B, and the left part of C or D cannot pack the item s' with side length $\ell' < \sqrt{2}/6$. From Lemma 4, the total area of free bricks in A, B, and the left part of C or D is less than $|S(s')| = \sqrt{2}/18$. From Lemma 5, the total occupied area in this bin is at least

$$\frac{|A| + |B| + |C|/2 - |S(s')|}{2\sqrt{2}} + {\ell'}^2 \geq \frac{|A| + |B| + |C|/2}{2\sqrt{2}} = 5/36.$$

By adding the contribution from the item s, the amortized occupation in this bin is at least

$$5/36 + (1 - \sqrt{2}/3)^2 - 1/4 = (10 - 6\sqrt{2})/9.$$

– Else, if packing s on the right-bottom corner of the bin overlaps with a small item s' with side length $\ell' < 1/3$ in E or F, similar to the above analysis, A, B, C and D cannot pack item s'. From Lemma 5, the total occupied area in this bin is at least

$$\frac{|A| + |B| + |C| + |D| - |S(s')|}{2\sqrt{2}} \geq 2/9.$$

Combining the above cases, the amortized occupation in those y bins is at least $(10 - 6\sqrt{2})/9$. □

A.5 The proof for Lemma 5

Proof. Now we consider the amortized occupation in the remaining z bins. W.l.o.g., assume that the j-th bin belongs to those z bins, and it is closed by the packing of an item s with side length ℓ.

– If $\ell \leq 1/3$, i.e., s is a small item. From the packing strategy, since s cannot be packed into this bin, bricks A, B, C, D, E, and F are all occupied by some items and cannot pack s. From Lemma 4 and Lemma 5, similar to previous analysis, the total occupation in this bin is at least

$$\frac{|A| + |B| + |C| + |D| + |E| + |F| - |S(s)|}{2\sqrt{2}} \geq 5/18$$

– Else, $1/3 < \ell \leq 1/2$, i.e., s is a middle item.
  - If there are 3 middle items packed in this bin, the occupied area is at least $1/3$.
  - Else, if there are 2 middle items packed in this bin,
    * if the side length ℓ' of one middle item s' is no less than $\sqrt{2}/3$, the amortized occupation is at least $(1/3)^2 + \ell'^2 - (\ell'^2 - 1/9)/2 = \ell'^2/2 + 1/6 \geq (\sqrt{2}/3)^2/2 + 1/6 = 5/18$.
    * else, two corners of this bin are occupied by middle items, and two bricks congruent with $(1/3, \sqrt{2}/3)$ are available for small items. Since the middle item s cannot be packed in this bin, all four corners of this bin are occupied by items. From Lemma 4 and Lemma 5, the occupied area of the small items in this bin is at least $1/18$. Since the contribution of each middle item is at least $1/9$, thus, in this case, the amortized occupation is at least $5/18$.
  - Else, if there is one middle item s' with side length ℓ' packed in this bin, this item must be packed either in the left-bottom corner of this bin, or in the right-bottom corner of this bin. Otherwise, the order A, B, C, D, E, and F for packing small items is violated.

Lemma 6. *If there is a middle item s' with side length ℓ' packed in the left-bottom corner of this bin, the amortized occupation in this bin is at least* $1/4$.

Proof. In this case, packing s in the right-bottom corner overlaps with some packed small item in E. Thus, the total area of small items in this bin is at least $1/9$.

* If $\ell' > \sqrt{2}/3$, the amortized occupation in this bin is at least

$$1/9 + {\ell'}^2 - ({\ell'}^2 - 1/9)/2 = {\ell'}^2/2 + 1/6 \geq 5/18.$$

* Else, s is packed within C and D, the contribution in this bin is at least $1/9$.
  · If packing s into this bin overlaps with a small item s'' in E whose side length ℓ'' satisfies $\sqrt{2}/6 \leq \ell'' \leq 1/3$, the occupied area in $A + B$ is at least $1/18$. In this case, $\ell + \ell'' > 2/3$. The amortized occupation in this bin is at least

$$\begin{aligned} & 1/18 + {\ell''}^2 + (\ell^2 - 1/9)/2 + {\ell'}^2 - ({\ell'}^2 - 1/9)/2 \\ = & 1/18 + {\ell''}^2 + \ell^2/2 + {\ell'}^2/2 \\ \geq & {\ell''}^2 + (2/3 - \ell'')^2/2 + {\ell'}^2/2 + 1/18 \\ = & 3(\ell'' - 4/9)^2/2 + {\ell'}^2/2 + 11/54 \\ \geq & 5/18 \qquad (\text{since } \sqrt{2}/6 \leq \ell'' \leq 1/3) \end{aligned}$$

  · If packing s into this bin overlaps with a small item s'' in E whose side length ℓ'' satisfies $1/6 \leq \ell'' \leq \sqrt{2}/6$, the occupied area in $A + B$ is at least $1/12$. In this case, $\ell + \ell'' > 2/3$. The amortized occupation in this bin is at least

$$\begin{aligned} & 1/12 + {\ell''}^2 + (\ell^2 - 1/9)/2 + {\ell'}^2 - ({\ell'}^2 - 1/9)/2 \\ = & 1/12 + {\ell''}^2 + \ell^2/2 + {\ell'}^2/2 \\ \geq & (2/3 - \ell)^2 + \ell^2/2 + {\ell'}^2/2 + 1/12 \\ = & 3(\ell - 4/9)^2/2 + {\ell'}^2/2 + 25/108 \\ \geq & 31/108 \end{aligned}$$

  · If packing s into this bin overlaps with a small item s'' in the left-bottom part of E and the side length ℓ'' satisfies $\sqrt{2}/12 \leq \ell'' < 1/6$, the occupied area in A, B, and the left-upper part of E is at least $1/9$. In this case, $\ell + \ell'' > 1 - \sqrt{2}/3$. The amortized occupation in this bin is at least

$$\begin{aligned} & 1/9 + {\ell''}^2 + (\ell^2 - 1/9)/2 + 1/9 \\ = & {\ell''}^2 + \ell^2/2 + 1/6 \\ > & {\ell''}^2 + (1 - \sqrt{2}/3 - \ell'')^2/2 + 1/6 \\ = & 3(\ell'' - (1 - \sqrt{2}/3)/3)^2/2 - 3((\sqrt{2}/3 - 1)/3)^2/2 + 7/9 - \sqrt{2}/3 \\ \geq & 0.2599 \end{aligned}$$

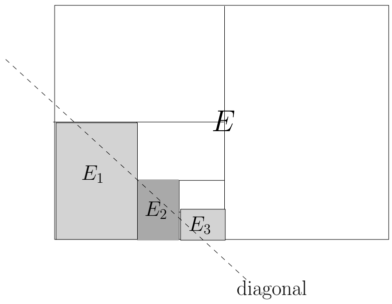

Fig. 6. overlap in brick E

· If packing s into this bin overlaps with a small item s'' with side length $\ell'' < \sqrt{2}/12$,
Case 1: the overlap happens in E_1 as shown in Figure. 6. The total occupation of small items in this bin is at least $1/8$. In this case, $\ell > 1 - \sqrt{2}/3 - \sqrt{2}/12$, thus, the contribution from s is $\ell^2/2 - 1/18 \geq 0.0288$. Therefore, the amortized occupation in this bin is at least $0.288 + 1/8 + 1/9 = 0.2649$.
Case 2: the overlap happens in E_2 as shown in Figure. 6. The total occupation of small items in this bin is at least $78/576$. In this case, $\ell > 1 - \sqrt{2}/3 - \sqrt{2}/12 - \sqrt{2}/24 \approx 0.3518$, and the contribution from s is at least $0.3518^2/2 - 1/18 = 0.0063$. Thus, the amortized occupation in this bin is at least $0.0063 + 78/576 + 1/9 = 0.2529$.
Case 3: Note that there is a brick E_3' in the right-top of this bin, and E_3 and E_3' are congruent bricks. In the packing strategy, if there is small items with side length no more than $1/24$ to be packed into E_3, we use E_3' to pack this item. Thus, if E_3 contains some item, the side length of this item must be strictly larger than $1/24$, and the total occupation of small items in this bin is at least $5/36$. If some small items are packed in the right part of E, the occupation of small items is strictly larger than $5/36$. Thus, the amortized occupation in this bin is at least $5/36 + 1/9 = 1/4$.
□

Lemma 7. *If there is a middle item packed in the right-bottom corner of this bin, the amortized occupation in this bin is at least 19/72.*

Proof. In this case, some small item must be packed in C. From Lemma 4 and Lemma 5, the total occupation of small items in this bin is at least $1/9$.

∗ If packing s into this bin overlaps with a small item s'' in C whose side length ℓ'' satisfies $\sqrt{2}/6 \leq \ell'' \leq 1/3$, the occupied area in $A + B$ is at least $1/18$. In this case, $\ell + \ell'' > 2/3$. The amortized occupation in this bin is at least

$$\begin{aligned}
&1/18 + \ell''^2 + (\ell^2 - 1/9)/2 + \ell'^2 - (\ell'^2 - 1/9)/2 \\
&= 1/18 + \ell''^2 + \ell^2/2 + \ell'^2/2 \\
&\geq \ell''^2 + (2/3 - \ell'')^2/2 + \ell'^2/2 + 1/18 \\
&= 3(\ell'' - 4/9)^2/2 + \ell'^2/2 + 11/54 \\
&\geq 5/18 \qquad (\text{since } \sqrt{2}/6 \leq \ell'' \leq 1/3)
\end{aligned}$$

* If packing s into this bin overlaps with a small item s'' in C whose side length ℓ'' satisfies $1/6 \leq \ell'' \leq \sqrt{2}/6$, the occupied area in $A+B$ is at least $1/12$. In this case, $\ell + \ell'' > 2/3$. The amortized occupation in this bin is at least

$$\begin{aligned}
&1/12 + \ell''^2 + (\ell^2 - 1/9)/2 + \ell'^2 - (\ell'^2 - 1/9)/2 \\
&= 1/12 + \ell''^2 + \ell^2/2 + \ell'^2/2 \\
&\geq (2/3 - \ell)^2 + \ell^2/2 + \ell'^2/2 + 1/12 \\
&= 3(\ell - 4/9)^2/2 + \ell'^2/2 + 25/108 \\
&\geq 31/108
\end{aligned}$$

* If packing s into this bin overlaps with a small item s'' in the left-bottom part of C whose side length ℓ'' satisfies $\ell'' < 1/6$, the occupied area of small items in this bin is at least $1/8$. In this case, $\ell + \ell' > 1 - \sqrt{2}/6$. The amortized occupation in this bin is at least

$$\begin{aligned}
&1/8 + (\ell^2 - 1/9)/2 + \ell'^2 - (\ell'^2 - 1/9)/2 \\
&= 1/8 + \ell^2/2 + \ell'^2/2 \\
&\geq 1/8 + (\ell + \ell')^2/4 \\
&\geq 1/8 + (1 - \sqrt{2}/6)^2/4 \\
&= (14 - 3\sqrt{2})/36
\end{aligned}$$

* If packing s into this bin overlaps with a small item s'' in the right-bottom part of C whose side length ℓ'' satisfies $\ell'' < 1/6$, the occupied area of small items in this bin is at least $11/72$. The contribution of packed middle item s' is at least $1/9$. Thus, the amortized occupation in this bin is at least $11/72 + 1/9 = 19/72$.

$\square$

- Else, if there is no middle item in this bin, from the order of packing small items, we can say that A, B, C, D, and E are all occupied by some small items. Similar to the proof in Lemma 6, the amortized occupation in this bin is at least $1/4$.

Combine the above cases, the amortized occupation in each of those z bins is at least $1/4$. $\square$